Math 4650
Topic 3 – Infinite Series

Def: Suppose we have a sequence $(a_n)_{n=1}^{\infty}$ and we want to make an infinite series out of it.

We define the partial sums by

$$s_k = a_1 + a_2 + a_3 + \dots + a_k$$

for $k \geq 1$.

That is,

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

and so on.

If $\lim_{k \to \infty} s_k$ exists and equals L, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty} a_n = L$.

If $\lim_{k \to \infty} s_k$ does not exist then we say that $\sum_{n=1}^{\infty} a_n$ diverges.

Note: The series can start at other numbers other that $n=1$, such as $\sum_{n=3}^{\infty} a_n = a_3 + a_4 + a_5 + \dots$

In that case just the starting point of S_k.
For example, for above we could do

$S_3 = a_3, \; S_4 = a_3 + a_4, \; S_5 = a_3 + a_4 + a_5, \ldots$

<u>Ex:</u> Consider $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

Let's calculate some partial sums.

k	$S_k = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}, \; k \geq 0$
0	$S_0 = 1$
1	$S_1 = 1 + \frac{1}{2} = 1.5$
2	$S_2 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$
3	$S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$
4	$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375$
$\vdots$	$\vdots$
50	$S_{50} \approx 1.\underbrace{999\ldots99}_{\text{fifteen 9's}}11182$
$\vdots$	$\vdots$
100	$S_{100} \approx 1.\underbrace{999\ldots99}_{\text{thirty 9's}}11139\ldots$

It seems that $\displaystyle\lim_{k \to \infty} S_k = 2$

so, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$

We will see why this is true, but we will look at $\sum_{n=0}^{\infty} r^n$

We will need this result about sequences.

Theorem: Let $r \in \mathbb{R}$.
If $|r|<1$, then $\lim_{n\to\infty} r^n = 0$.

proof: HW 3. $\blacksquare$

Ex: (Geometric series)

We are interested in the series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \dots$$

Note there is some ambiguity in the notation when $r=0$. What is 0^0?

When $r=0$ the sum above is defined to be

$$\sum_{n=0}^{\infty} 0^n = 1 + 0 + 0^2 + 0^3 + \dots$$

Let's now assume that $|r|<1$.

We want to look at the partial sums s_k.

Note that

$$\begin{aligned} s_k(1-r) &= (1+r+r^2+\dots+r^k)(1-r) \\ &= 1+r+r^2+\dots+r^k \\ &\quad -r-r^2-\dots-r^k-r^{k+1} \\ &= 1-r^{k+1} \end{aligned}$$

Thus,

$$s_k = \frac{1-r^{k+1}}{1-r}$$

Thus, if $|r|<1$ then

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r} = \frac{1-0}{1-r} = \frac{1}{1-r}$$

Therefore, if $|r|<1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Ex: $$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$\boxed{r = 1/2,\ |r|<1}$

Ex: Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

We can use partial fractions to get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$n=-1$: $1 = A(0) + B(-1)$

$B = -1$

$n=0$: $1 = A(1) + B(0)$

$1 = A$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and

$$s_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2}$$

$$s_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) = 1 - 1/3$$

$$s_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

In general for $k \geq 1$ we get

$$s_k = 1 - \frac{1}{k+1}$$

Proof by induction.

$n=1$: $s_1 = 1 - \frac{1}{1+1}$

Assume $s_k = 1 - \frac{1}{k+1}$.

Then, $s_{k+1} = s_k + \left(\frac{1}{k+1} - \frac{1}{(k+1)+1}\right)$

$= \left(1 - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{(k+1)+1}\right)$

$= 1 - \frac{1}{(k+1)+1}$ $\blacksquare$

Thus,

$$\lim_{k \to \infty} s_k = 1 - 0 = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

This series is called a "telescoping series" because of how the terms cancel each other out in the partial sums.

Theorem: (Divergence Test)

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

[Thus, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges]

Proof:

Suppose $\sum a_n$ converges to L.

Then $\lim_{k\to\infty} s_k = L$.

Since

$$a_n = (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1})$$

$$= s_n - s_{n-1}$$

We get that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$$

$\blacksquare$

Ex: $\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

diverges because $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + 1/n} = \frac{1}{1+0}$

$= 1 \neq 0$

Ex: The harmonic series is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Note that $\lim_{n\to\infty} \frac{1}{n} = 0$.

So we can't use the divergence test.

However, it turns out that this series will diverge.

Let's consider the partial sums

$$s_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

In order for $\lim_{k\to\infty} s_k$ to exist we would need that (s_k) is a Cauchy sequence.

We will show this is not the case.

If $m > n$, then

$$\begin{aligned} |s_m - s_n| &= \left| \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) \right| \\ &= \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \right| \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \\ &> \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m} \\ &= \frac{m-n}{m} \\ &= 1 - \frac{n}{m} \end{aligned}$$

In particular, if $m = 2n$ then

$$|S_{2n} - S_n| = 1 - \frac{1}{n} > \frac{1}{2}$$

This implies that $(s_k)_{k=1}^{\infty}$ is not Cauchy.

Why?

If (s_k) was cauchy then there would exist $N > 0$ where if $m, n \geq N$, then

$$|s_m - s_n| < \frac{1}{2}$$

But picking $n \geq N$ and $m = 2n \geq N$ we get $|s_m - s_n| = |s_{2n} - s_n| > \frac{1}{2}$.

Thus, (s_k) diverges and so does $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\blacksquare$

Lemma: Suppose (a_n) is a monotonically increasing sequence. If there exists a subsequence (a_{n_k}) that is bounded from above, then (a_n) is a bounded sequence.

proof:

We know that $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \dots$

So, (a_n) is bounded below by a_1.

Suppose there exists a subsequence

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \dots$$

that is bounded from above by M, where $n_1 < n_2 < n_3 < \dots$.

That is, $a_{n_k} < M$ for all n_k.

Let a_n be in the original sequence (a_n).

Pick some n_{k_0} such that $a_{n_{k_0}}$ is in the subsequence and $n < n_{k_0}$.

Then, $a_n \leq a_{n_{k_0}} < M$

sequence is monotonically increasing

Thus, $a_1 \leq a_n < M$. So, (a_n) is a bounded sequence. $\blacksquare$

Ex: (p-series with $p>1$).

We will show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges if $p > 1$.

proof: Let S_k be the k-th partial sum of the series, that is

$$S_k = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$$

Then,

$$\frac{1}{1^p} < \frac{1}{1^p} + \frac{1}{2^p} < \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} < \cdots$$

So,

$$S_1 < S_2 < S_3 < S_4$$

That is, $(S_k)_{k=1}^{\infty}$ is a monotonically increasing sequence.

The plan will be to come up with a subsequence that is bounded from above. Then, by the lemma, $(S_k)_{k=1}^{\infty}$ will be bounded.

Then, by the monontone convergence theorem the sequence $(s_k)_{k=1}^{\infty}$ will converge.

Now we construct the subsequence.

First term:

$k_1 = 2^1 - 1 = 1$

$s_{k_1} = s_1 = \frac{1}{1^p} = 1.$

Second term:

Set $k_2 = 2^2 - 1 = 3.$

Then,

$$s_{k_2} = s_3 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right)$$

$2^p < 3^p$ since $p > 1$

$$= 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

Third term:

Set $k_3 = 2^3 - 1 = 7.$

Then,

$$S_{k_3} = S_7 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$< 1 + \frac{1}{2^{p-1}} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2$$

j-th term:

In general, let $k_j = 2^j - 1$.

Let $r = \frac{1}{2^{p-1}}$.

Then,

$$S_{k_j} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{7^p}\right)$$

$$+ \cdots + \left(\underbrace{\frac{1}{(2^j - 2^{j-1})^p}} + \cdots + \frac{1}{(2^j - 2)^p} + \frac{1}{(2^j - 1)^p}\right)$$

$$\boxed{2^j - 2^{j-1} = 2^{j-1}}$$

$$< 1 + 2\cdot\left(\frac{1}{2^p}\right) + 2^2\left(\frac{1}{(2^2)^p}\right) + \cdots + 2^{j-1}\left(\frac{1}{(2^{j-1})^p}\right)$$

$$= 1 + \left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2^2}\right)^{p-1} + \cdots + \left(\frac{1}{2^{j-1}}\right)^{p-1}$$

$$= 1 + r + r^2 + \cdots + r^{j-1}$$

$$= \frac{1 - r^j}{1 - r}$$

$$< \frac{1}{1-r}$$

Thus, we have a bounded subsequence $(s_{k_j})_{j=1}^{\infty}$

As described above we get that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

$\boxed{/\!/}$

Note: It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020569\ldots \leftarrow \text{Apéry's constant}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278\ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$ where B_{2k} is the $2k$-th Bernoulli number

(These are values of the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$)

Theorem: (Comparison test)

Suppose that $0 \leq a_n \leq b_n$ for all $n \geq K$ for some fixed $K > 0$.

Then:

① If $\sum b_n$ converges, then $\sum a_n$ converges.

② If $\sum a_n$ diverges, then $\sum b_n$ diverges.

---

proof:

② is the contrapositive of ①, so we just need to prove ①.

Suppose that $\sum b_n$ converges.

Let (s_k) be the partial sums of $\sum b_n$ and (t_k) be the partial sums of $\sum a_n$.

That is,

$$s_k = b_1 + b_2 + b_3 + \dots + b_k$$

$$t_k = a_1 + a_2 + a_3 + \dots + a_k$$

Since $\sum b_n$ converges we know that (s_k) converges, so it's a Cauchy sequence.

Let $\varepsilon > 0$.
Then there exists $N > K$ where if $m \geq n \geq N$, then $|s_m - s_n| < \varepsilon$.

Therefore if $m \geq n \geq N$, then

$$
\begin{aligned}
|t_m - t_n| &= |a_1 + a_2 + \cdots + a_m - a_1 - a_2 - \cdots - a_n| \\
&= |a_{n+1} + a_{n+2} + \cdots + a_m| \\
&= a_{n+1} + a_{n+2} + \cdots + a_m \\
&\leq b_{n+1} + b_{n+2} + \cdots + b_m \\
&= |b_{n+1} + b_{n+2} + \cdots + b_m| \\
&= |b_1 + b_2 + \cdots + b_m - b_1 - b_2 - \cdots - b_n| \\
&= |s_m - s_n| \\
&< \varepsilon
\end{aligned}
$$

Thus, (t_k) is a Cauchy sequence.
So, (t_k) converges and so does $\sum a_n$. $\blacksquare$

Ex: Since $0 < \frac{1}{n^2+n} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the comparison test we get that $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Ex: (p-series)

Suppose $0 < p < 1$.

If $n \in \mathbb{N}$, then $n^p \leq n$.

So, if $n \in \mathbb{N}$, then $\frac{1}{n} \leq \frac{1}{n^p}$.

Thus, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test we know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Theorem: (Alternating series test)
Let (a_n) be a monotonically decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$.

Then, the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

converges.

---

proof: Let (s_k) be the partial sums, that is

$$s_k = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{k+1} a_k$$

First we look at the subsequence (s_{2k}).

Since (a_n) is monotonically decreasing for all $n \geq 1$, we know that $a_n - a_{n+1} \geq 0$

So,

$$\begin{aligned} s_{2k} &= (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2k-1} - a_{2k}) \\ &< (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2k-1} - a_{2k}) + (a_{2k+1} - a_{2k+2}) \\ &= s_{2(k+1)} \end{aligned}$$

So, $(s_{2k})_{k=1}^{\infty}$ is monotonically increasing.

Also,

$$S_{2k} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} - \dots - \underbrace{(a_{2k-2} - a_{2k-1})}_{\geq 0} - \underbrace{a_{2k}}_{>0}$$
$$\leq a_1$$

Thus, since $(S_{2k})_{k=1}^{\infty}$ is monotonically increasing and bounded from above.

By the monotone convergence theorem

$\lim_{k\to\infty} S_{2k} = L$ for some $L \in \mathbb{R}$.

We now show that $\lim_{k\to\infty} S_k = L$.

Let $\varepsilon > 0$

Since $\lim_{k\to\infty} S_{2k} = L$ there exists $N_1 > 0$ where if $k \geq N_1$ then $|S_{2k} - L| < \frac{\varepsilon}{2}$

Since $\lim_{k\to\infty} a_k = 0$, there exists $N_2 > 0$ where if $k \geq N_2$ then $|a_{2k+1}| = |a_{2k+1} - 0| < \frac{\varepsilon}{2}$.

If $k \geq \max\{N_1, N_2\}$ then

$$|S_{2k+1} - L| = |S_{2k} + a_{2k+1} - L|$$

$$\leq |s_{2k} - L| + |a_{2k+1}|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, for $k \geq \max\{N_1, N_2\}$ we get both

$|s_{2k} - L| < \frac{\varepsilon}{2} < \varepsilon$ and $|s_{2k+1} - L| < \varepsilon$

Note that:

$k \geq \max\{N_1, N_2\}$ iff $2k \geq 2\max\{N_1, N_2\}$ or $2k+1 \geq 2\max\{N_1, N_2\} + 1$

Therefore, if $\ell \geq 2\max\{N_1, N_2\} + 1 > 2\max\{N_1, N_2\}$,

then $|s_\ell - L| < \varepsilon$.

So, $(s_k)_{k=1}^{\infty}$ converges to L. $\blacksquare$

Ex: Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $0 < \frac{1}{n+1} < \frac{1}{n}$ for all $n \geq 1$ we get that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges.